Problem Set IV: The Finite Calculus

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Let's develop a *discrete* analogue to calculus that will allow us to evaluate sums and recurrences more robustly.

Given a function $f : \mathbb{Z} \to \mathbb{R}$ (note that we have no other constraints!), let's define three analogues to concepts from calculus:

1. We define the *difference of f*, $\Delta f : \mathbb{Z} \to \mathbb{R}$, such that $(\Delta f)(x) := f(x+1) - f(x)$ **[analogue of the derivative]**
2. We define the *anti-difference of f*, written as $\sum f(x)\delta x$, as the class of functions F such that $\Delta F = f$. **[analogue of the anti-derivative]**
3. We define the *definite sum of f from a to b*, by $\sum_a^b f(x)\delta x := \sum_{k=a}^{b-1} f(k)$ (in particular, for $b \leq a$ we define this to be 0). **[analogue of the definite integral]**

Finally, we define the *shift of f*, $\mathbf{E}f : \mathbb{Z} \to \mathbb{R}$, such that $(\mathbf{E}f)(x) := f(x+1)$ (note – this has no analogue in usual calculus). It's straightforward to check that these behave like derivatives and integrals in many ways:

- Given $f, g : \mathbb{Z} \to \mathbb{R}$, we have that $\Delta(f+g) = \Delta f + \Delta g$, and also, for any $\alpha \in \mathbb{R}$, $\Delta(\alpha f) = \alpha(\Delta f)$. Therefore, Δ is a linear operator. The same is true of anti-differences (though one must be careful, since we usually speak of *an* anti-difference instead of *the* anti-difference).
- Multiplication is unfortunately not as well-behaved as usual:

$$\begin{aligned}
\Delta(fg)(n) &= (fg)(n+1) - (fg)(n) \\
&= f(n+1)g(n+1) - f(n)g(n) \\
&= f(n+1)g(n+1) - f(n+1)g(n) + f(n+1)g(n) - f(n)g(n) \\
&= f(n+1)(g(n+1) - g(n)) + g(n)(f(n+1) - f(n)) \\
&= (\mathbf{E}f)(n)(\Delta g)(n) + g(n)(\Delta f)(n)
\end{aligned}$$

And therefore, $\Delta fg = \Delta f \cdot g + \mathbf{E}f \cdot \Delta g$. Close enough, I guess.

Instructions: No need to solve them fully, just ponder them. We'll probably go over the solutions of a few of them during class. Have fun!

Problem 1. (The fundamental theorems of finite calculus, pt. I)
We are given $f : \mathbb{Z} \to \mathbb{R}$. Let $F : \mathbb{Z} \to \mathbb{R}$ be such that $F(t) := \sum_k^t f(x)\delta x$, for some

integer k. Verify that $\Delta F = f$, and therefore $F \in \sum f(x)\delta x$. This gives us a way to obtain anti-differences for free.

Problem 2. (The fundamental theorems of finite calculus, pt. II)
Let $f : \mathbb{Z} \to \mathbb{R}$, and let $F : \mathbb{Z} \to \mathbb{R}$ be *any* anti-difference of F (that is, $\Delta F = f$). Prove that $\sum_a^b f(x)\delta x = F(b) - F(a)$. This gives us a way to obtain definite sums for free if we happen to know an anti-difference.

Problem 3. (Falling powers)
Unfortunately, if we try to compute $\Delta(x^n)$, we obtain $nx^{n-1} +$ junk in $\Theta(x^{n-2})$, which means that the usual power rule doesn't work as expected. However, if we can define the *falling power*

$$x^{\underline{n}} := x(x-1)...(x-(n-1))$$

Verify that $\Delta x^{\underline{n}} = nx^{\underline{n-1}}$. Immediately, we get that $\frac{x^{\underline{n+1}}}{n+1}$ is an anti-difference of $x^{\underline{n}}$. Equipped with these facts, find the general formula for

$$\sum_{k=1}^{n} k^3$$

Problem 4. (Summation by parts)
Recall that we found that $\Delta fg = \Delta f \cdot g + \mathbf{E}f \cdot \Delta g$. Can you use this to obtain a discrete analog to integration by parts

$$\int uv' = uv - \int u'v$$

Furthermore, compute $\Delta(b^x)$ for a particular base b.

Try to obtain a closed-form formula for

$$\sum_{k=1}^{n} k2^k$$