Problem Set II: Solving Recurrences

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Suppose that we wanted to solve the recurrence (without caring for the initial values):

$$R_n = k_1 R_{n-1} + k_2 R_{n-2} + \dots + k_\ell R_{n-\ell} \text{ for fixed } k_1, \dots, k_\ell \in \mathbb{R}, n \geq \ell$$

We can make two very interesting observations:

1. If sequences $(f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty}$ satisfy this recurrence, then the sequence $(f_n + g_n)_{n=1}^{\infty}$ satisfies this recurrence.
2. If the sequence $(f_n)_{n=1}^{\infty}$ satisfies this recurrence, then for all $\alpha \in \mathbb{R}$, $(\alpha f_n)_{n=1}^{\infty}$ also satisfies this recurrence.

Formally, this means that the set of solutions to this recurrence form a vector space over $\mathbb{R}$. The observations give us a good plan: to find a solution to the given recurrence *with given initial values*, we just need to find any (or a few) solutions to the recurrence that *may have other initial values* but that we can manipulate using these operations into the initial values we want.

Claim. *If $\lambda \in \mathbb{R}$ has the property that $\lambda^\ell = k_1 \lambda^{\ell-1} + k_2 \lambda^{\ell-2} + \dots + k_\ell$ then $(\lambda^n)_{n=1}^{\infty}$ is a solution to the recurrence.*

Proof. Multiplying by $\lambda^{n-\ell}$ on both sides, we obtain the recurrence that the sequence must satisfy. ■

As a good example, recall that the Fibonacci sequence is defined through the recurrence $F_n = F_{n-1} + F_{n-2}$ (and initial values which we don't care about at the moment), which happens to be of the form above; therefore, the solutions $\varphi = \frac{1+\sqrt{5}}{2}, \psi = \frac{1-\sqrt{5}}{2}$ to the quadratic $\lambda^2 = \lambda + 1$ have the property that the sequences $(\varphi_1^n)_{n=1}^{\infty}$ and $(\varphi_2^n)_{n=1}^{\infty}$ satisfy the Fibonacci recurrence. However, note that:

$$\begin{cases} F_0 = 0 \text{ and } F_1 = 1 \\ \varphi^0 = 1 \text{ and } \varphi^1 = \varphi \\ \psi^0 = 1 \text{ and } \psi^1 = \psi \end{cases}$$

So, we're not done yet. Now, by the observations, for all $\alpha, \beta \in \mathbb{R}$, $(\alpha\varphi^n + \beta\psi^n)_{n=1}^{\infty}$ is a solution to the recurrence, meaning that if we find solutions α, β to the system of linear equations

$$\begin{cases} \alpha + \beta = 0 \\ \alpha\varphi + \beta\psi = 1 \end{cases}$$

then $(\alpha\varphi^n + \beta\varphi^n)_{n=1}^{\infty}$ will precisely be the Fibonacci sequence, because it has the correct initial values and it satisfies the recurrence.

Instructions: No need to solve them fully, just ponder them. We'll probably go over the solutions of a few of them during class. Have fun!

Problem 1. (Closed form of Fibonacci, pt. I)
Solve the above system of equations to prove that

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}} \text{ for all integers } n \geq 0.$$

Problem 2. (Closed form of Fibonacci, pt. II)
We can observe the following two things:

1. For all $n \in \mathbb{Z}_{\geq 0}$, F_n is an integer.
2. In general, $E(n) = \left|\frac{\psi^n}{\sqrt{5}}\right|$ seems to be quite small.

Try to find a bound for $E(n)$ that allows you to find an even simpler closed form expression for F_n (using the one found in Problem 1).

Problem 3. (Limitations of the method)
The method is actually severely limited. Consider the sequence $R_n = n$, which satisfies the recurrence $R_n = R_{n-1} + 1$ for $n \geq 1$. This is not a recurrence of the desired form, but we can make it so by subtracting $R_{n-1} = R_{n-2} + 1$ from it, to obtain $R_n - R_{n-1} = R_{n-1} - R_{n-2}$, or equivalently $R_n = 2R_{n-1} - R_{n-2}$. What happens if we try to apply the method to this recurrence? For what recurrences can we guarantee that the sequence will work well?